

Reading: Read [Gui24]. For more on chromatic homotopy theory, see [Bea19] or [Rav23].

- (1) Let X be a spectrum and $e: X \rightarrow X$ an *idempotent* map: $e \circ e = e$. Construct a spectrum X_e such that for all $n \in \mathbb{Z}$,

$$\pi_n(X_e) = \text{im}(\pi_n X \xrightarrow{e_*} \pi_n X).$$

Thus, idempotent maps have “images” in $\text{ho}(\mathcal{S}p)$, even though it is not an abelian category.

SOLUTION: First, recall the previously discussed facts that sequential homotopy colimits of spectra are computed level-wise and that homotopy groups commute with sequential colimits. Let $X_e = \text{hocolim}(X \xrightarrow{e} X \xrightarrow{e} \dots)$. It follows that

$$\begin{aligned} \pi_n X_e &\cong \text{colim}_k \pi_{n+k}(X_e)_k \\ &\cong \text{colim}_k \pi_{n+k} \text{hocolim}(X_k \xrightarrow{e_k} X_k \xrightarrow{e_k} \dots) \\ &\cong \text{colim}_k \text{colim}(\pi_{n+k} X_k \xrightarrow{e_{k*}} \pi_{n+k} X_k \xrightarrow{e_{k*}} \dots) \\ &\cong \text{colim}(\text{colim}_k \pi_{n+k} X_k \xrightarrow{e_*} \text{colim}_k \pi_{n+k} X_k \xrightarrow{e_*} \dots) \\ &\cong \text{colim}(\pi_n X \xrightarrow{e_*} \pi_n X_k \xrightarrow{e_*} \dots) \cong \text{im}(e_*) \end{aligned}$$

where the last isomorphism is a standard fact about abelian groups.

- (2) If A is a commutative ring, a regular sequence $a_1, a_2, \dots$ is a sequence of elements of A such that multiplication by a_n is injective on $A/(a_1, \dots, a_{n-1})$.

If R is a commutative ring spectrum and $x_1, x_2, \dots$ is a regular sequence in $\pi_* R$, show that

$$\pi_*(R/(x_1, x_2, \dots)) \cong (\pi_* R)/(x_1, x_2, \dots),$$

where $R/(x_1, x_2, \dots)$ is the homotopy colimit of the spectra $R/(x_1, x_2, \dots, x_n)$. Recall that R/x_1 is the homotopy cofiber of $x_1: \Sigma^{k_1} R \rightarrow R$, and $R/(x_1, x_2)$ is the homotopy cofiber of $x_2: \Sigma^{k_2}(R/x_1) \rightarrow R/x_1$, etc.

- (3) Recall that the spectra $MU_{(p)}$ and BP with

$$\begin{aligned} \pi_* MU_{(p)} &= \mathbb{Z}_{(p)}[x_1, x_2, \dots], & |x_i| &= 2i, \\ \pi_* BP &= \mathbb{Z}_{(p)}[v_1, v_2, \dots], & |v_i| &= 2(p^i - 1). \end{aligned}$$

Assuming that $MU_{(p)}$ splits as a wedge sum of shifts of copies of BP , show that there must be infinitely many copies of BP in this wedge sum.

- (4) Use the chromatic fracture square to show that if X is $E(1)$ -local, then X/p is $K(1)$ -local.

REFERENCES

- [Bea19] Agnes Beaudry. An introduction to chromatic homotopy theory. eCHT Minicourse, <https://s.wayne.edu/echt/echt-minicourses/>, 2019.
- [Gui24] Bert Guillou. Class Notes for Chromatic Homotopy Theory, Math 751. <https://www.ms.uky.edu/~guillou/S24/751Notes.pdf>, 2024.

[Rav23] Doug Ravenel. The background and motivation for the telescope conjecture. eCHT Minicourse, <https://s.wayne.edu/echt/echt-minicourses/>, 2023.

Credit for all problems to Bert Guillou.