

ESSAY QUESTION

Using examples from class (or elsewhere), write 1-2 pages of TeX^* to argue for or against the following claim:

Spectra are more fundamental to algebraic topology than spaces.

Cite any resources you use.

PROBLEMS

- (1) Define a category $\mathcal{Z}\mathbf{p}$ of “zpectra” whose objects are $\mathbb{Z}$ -indexed sequences of spaces $\{X_n\}_{n \in \mathbb{Z}}$ together with structure maps $\sigma_i: \Sigma X_i \rightarrow X_{i+1}$ for all $i \in \mathbb{Z}$. A *morphism of zpectra* $f: X \rightarrow Y$ is a sequence of continuous maps $f_i: X_i \rightarrow Y_i$ that commute with the structure maps of X and Y .

The *ztable homotopy groups* of a zpectrum X are defined by $\pi_k X := \text{colim}_{n \in \mathbb{Z}} \pi_{n+k} X_n$. A *ztable equivalence* of zpectra is a map that induces isomorphisms on stable homotopy groups.

Prove that the homotopy category of zpectra and ztable equivalences is equivalent to $\text{ho}(\mathcal{S}\mathbf{p})$.

- (2) Let $X \xrightarrow{f} Y \rightarrow Z$ be a cofiber sequence such that f is zero in $\text{ho}(\mathcal{S}\mathbf{p})$. Show that $Z \simeq Y \vee \Sigma X$.
- (3) Let $\widehat{\mathcal{S}\mathbf{p}}$ be any symmetric monoidal category of spectra. Given a spectrum X , define $T(X) := \bigvee_{n \geq 0} X^{\wedge n}$.
- (a) Prove that $T(X)$ is an associative ring spectrum.
 - (b) Prove that the functor $T: \widehat{\mathcal{S}\mathbf{p}} \rightarrow \text{Mon}(\widehat{\mathcal{S}\mathbf{p}})$ is left adjoint to the forgetful functor $U: \text{Mon}(\widehat{\mathcal{S}\mathbf{p}}) \rightarrow \widehat{\mathcal{S}\mathbf{p}}$, where $\text{Mon}(\widehat{\mathcal{S}\mathbf{p}})$ is the category of monoids in $\widehat{\mathcal{S}\mathbf{p}}$, i.e. associative ring spectra.
- (4) Let X be a symmetric spectrum. Recall that the symmetric spectrum $\text{sh}^1 X$ is the symmetric spectrum with $(\text{sh}^1 X)_n = X_{1+n}$, where Σ_n acts on X_{1+n} as the subgroup of Σ_{1+n} consisting of those permutations of $\{1, \dots, n+1\}$ leaving 1 fixed.
- (a) Construct a symmetric spectrum $(\text{sh}^{-1} X)$ with n -th space $(\Sigma_n)_+ \wedge_{\Sigma_{n-1}} X_{n-1}$.
 - (b) Show that sh^{-1} is a functor on symmetric spectra which is left adjoint to sh^1 .
- (5) If R is a commutative ring spectrum and $x \in \pi_n R$, let $R[x^{-1}]$ be the homotopy colimit of

$$R \xrightarrow{x} \Sigma^{-n} R \xrightarrow{x} \Sigma^{-2n} R \xrightarrow{x} \dots,$$

where by abuse of notation we write $x: \Sigma^n R \rightarrow R$ for any suspension/desuspension of the map

$$\Sigma^n R = S^n \wedge R \xrightarrow{x \wedge \text{id}_R} R \wedge R \xrightarrow{\mu} R.$$

Show that $\pi_*(R[x^{-1}]) = (\pi_* R)[x^{-1}]$.

*Or 1-2 pages single spaced in a word processor, about 500-1000 words. I am not a stickler for essay length.