Power ops + global algebra

Context:

$$\begin{tikzcd} S^0 \arrow[r] \arrow[dr, bend left] \arrow[dd] & H\mathbb{F}_p \\ & \vdots \\ & E_3 \\ & E_2 \\ KU \arrow[r] & KU^\wedge_p = E_1 \end{tikzcd}$$

$E_3 \rightsquigarrow$ values on finite gps give rise to objects in global algebra

ht 1

The complex rep ring

The basics: A G-rep is a finite dim $\mathbb{C}$-vs V w/ an action of G $\leftrightsquigarrow$ $G \xrightarrow{f} \mathrm{Aut}(V) \cong GL_n(\mathbb{C})$

$\leftrightsquigarrow$ fd $\mathbb{C}[G]$-mod

A map $V \to W$ of G-reps is an equivariant lin map.

Tensor product: $V \otimes_{\mathbb{C}} W \circlearrowleft G$ acts diagonally.

Let $[V]$ be the iso class of V

Define $[V] + [W] = [V \oplus W]$ $\qquad 0 = [\mathbb{C}^0]$

$[V][W] = [V \otimes W]$ $\qquad 1 = [\mathbb{C}^1 \overset{\text{triv}}{\circlearrowleft} G]$

Let $RU(G)$ be the Gr. ring of iso classes of G-reps, $+, \cdot$

Schur's lemma: If V + W are irred G-reps then

1) if $V \ncong W$ and $f: V \to W$ is a map of G-reps then $f = 0$.
2) if $V \cong W$ any map $f: V \to W$ is mult by a scaler.

Ex. Assume A is abelian and V is an irred A-rep.

For $a \in A$, the action map $a: V \longrightarrow V$ is a map of A-reps.

Schur's lemma $\Rightarrow$ $av = cv$ for some $c \in \mathbb{C}$

So $\langle v \rangle$ is a sub rep of $V \Rightarrow \dim V = 1$.

Prop: Every G-rep is a sum of irreds in a $\overset{\text{Schur's lemma}}{\text{unique way}}$ so $RU(G)$ is additively a free $\mathbb{Z}$-mod w/ canonical basis given by the set of iso classes of irred G-reps.

Ex. $RU(e) \cong \mathbb{Z}$

Ex. $RU(C_n)$ $\quad C_n \overset{\rho}{\curvearrowright} \mathbb{C}$ by rotation by $\frac{2\pi i}{n}$

$C_n \overset{\rho\otimes\rho}{\curvearrowright} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ rotation by $\frac{4\pi i}{n}$.

$\rho^{\otimes n} = \text{triv}$ and these powers are all non iso so, if $x = [\rho]$, then

$$RU(C_n) \cong \mathbb{Z}[x]/x^n - 1$$

$\mathbb{C}\{C_n\} \circlearrowleft C_n$ the regular rep

$[\mathbb{C}\{C_n\}] = 1 + x + x^2 + \dots + x^{n-1}$.

<u>Restrictions + transfers</u>

let $f : H \to G$

$\mathrm{Res}_f : RU(G) \longrightarrow RU(H)$ is a ring map

$G \curvearrowright V \longmapsto H \xrightarrow{f} G \curvearrowright V$

$\mathrm{Tr}_f : RU(H) \longrightarrow RU(G)$ is an additive map

$H \curvearrowright V \longmapsto G \curvearrowright \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$

Ex. $e \overset{i}{\hookrightarrow} G$ $\quad \mathrm{Res}_i : RU(G) \longrightarrow RU(e) \cong \mathbb{Z}$

$V \longmapsto \dim V$

$\mathrm{Tr}_i : RU(e) \cong \mathbb{Z} \longrightarrow RU(G)$

$\mathbb{C} \longmapsto \mathbb{C}\{G\} \circlearrowleft G$

Ex. $f : H \twoheadrightarrow G$ $\quad \mathrm{Tr}_f([V]) = [V/\ker f]$

Interact in two important ways:

① Double coset formula: Given $H, K \leq G$

$$\mathrm{Res}^G_K \mathrm{Tr}^G_H = \sum_{KsH \in K\backslash G/H} \mathrm{Tr}^K_{K\cap sHs^{-1}} c_s \mathrm{Res}^H_{s^{-1}Ks \cap H}$$

② Frobenius Reciprocity:

$x \in RU(G)$ $y \in RU(H) \Rightarrow \mathrm{Tr}_f(\mathrm{Res}_f(x)y) = x\,\mathrm{Tr}_f(y)$.

$\Rightarrow$ im Tr_f is an ideal in $RU(G)$.

Ex. Assume $n = kd$

$$C_k \overset{f}{\hookrightarrow} C_n \overset{g}{\twoheadrightarrow} C_n/C_k \cong C_d$$

$$\begin{array}{ccc} \mathrm{Res}_f : RU(C_n) & \twoheadrightarrow & RU(C_k) \\ \cong & & \cong \\ \mathbb{Z}[x]/x^n-1 & & \mathbb{Z}[x]/x^k-1 \\ x & \longmapsto & x \end{array}$$

surjective

$\mathrm{Tr}_f : RU(C_k) \longrightarrow RU(C_n)$ determined by where $1 = [\mathbb{C}^1]$ goes because Res_f is surj.

$$\mathbb{C} \longmapsto \mathbb{C}[C_n] \otimes_{\mathbb{C}[C_k]} \mathbb{C} = \mathbb{C}\{C_n/C_k\}$$

$$\begin{array}{ccc} \mathrm{Res}_g : RU(C_d) & \longrightarrow & RU(C_n) \\ x = [\rho] & \longmapsto & x^k \\ \mathbb{C}\{C_d\} & \longmapsto & \mathbb{C}\{C_n/C_k\} \\ \| & & \\ 1 + x + x^2 + \dots + x^{d-1} & \longmapsto & 1 + x^k + x^{2k} + \dots + x^{(d-1)k}. \end{array}$$

Ex. $\mathrm{Tr}^e_{C_2} : RU(C_2) \longrightarrow \mathbb{Z}$

$$\begin{array}{ccc} 1 & \longmapsto & 1 \\ x & \longmapsto & 0 \end{array}$$

<u>Kunneth Iso</u>: Projections $G_1 \times G_2 \xrightarrow{\pi_i} G_i$

Restriction gives $\boxtimes : RU(G_1) \otimes RU(G_2) \longrightarrow RU(G_1 \times G_2)$

Fact: This is an iso for Rep rings.

<u>Character theory</u>: let $G/_{conj}$ the set of conj classes of elts in G.

let $\mathbb{Q}(\mu_\infty)$ be $\mathbb{Q}$ adjoin all roots of unity.

let $\rho : G \longrightarrow GL_n(\mathbb{C})$ be a rep.

then consider $tr(\rho(g))$

As $\rho(g)$ can be put in Jordan normal form and since g has finite order, there are roots of unity along diagonal

so $tr(\rho(g))$ is a sum of roots of unity

Since $tr(\rho(hgh^{-1})) = tr(\rho(g))$, get a map

$$RU(G) \xrightarrow{\chi} \mathrm{Fun}(G/_{conj}, \mathbb{Q}(\mu_\infty)) = Cl(G, \mathbb{Q}(\mu_\infty))$$

Fact: χ is an injective ring map and it induces an iso

$$\mathbb{Q}(\mu_\infty) \otimes RU(G) \xrightarrow{\cong} Cl(G, \mathbb{Q}(\mu_\infty)).$$

Even more: let $\hat{\mathbb{Z}} = \lim \mathbb{Z}/n \subseteq \prod \mathbb{Z}/n$ the profinite integers.

$$\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) = \hat{\mathbb{Z}}^\times = \mathrm{Aut}(\hat{\mathbb{Z}})$$

$$G/_{conj} = \hom_{cts}(\hat{\mathbb{Z}}, G)/_{conj} \circlearrowleft \hat{\mathbb{Z}}^\times$$

$$\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$$

$$\mathbb{Q} \otimes RU(G) \xrightarrow{\cong} Cl(G, \mathbb{Q}(\mu_\infty))^{\hat{\mathbb{Z}}^\times}$$

$$\mathbb{Z}/n \to \mathbb{Z}/d \qquad \mathbb{Z}/n \to \mathbb{Z}/k$$

$$\hom_{cts}(\hat{\mathbb{Z}}, G) \cong \hom(\mathbb{Z}/|G|, G)$$

Lecture 2

Ex: If $G = C_n$ then $\chi : RU(C_n) \longrightarrow Cl(C_n, \mathbb{Q}(\mu_\infty))$

$$x \longmapsto (1, \mu_n, \mu_n^2, \dots, \mu_n^{n-1})$$

so $\mathbb{Q}(\mu_\infty) \otimes RU(C_n) \cong \prod_{C_n} \mathbb{Q}(\mu_\infty)$

Important example: $G = \Sigma_m$

$$\chi : RU(\Sigma_m) \longrightarrow Cl(\Sigma_m, \mathbb{Q}(\mu_\infty))^{\hat{\mathbb{Z}}^\times}$$

$[\sigma] \in \Sigma_m/\mathrm{conj}$ determined by length of cycles in cycled decomp.

$\hat{\mathbb{Z}}^\times \curvearrowright \Sigma_m/\mathrm{conj}$ $\quad$ $l \in \hat{\mathbb{Z}}^\times \quad [\sigma] \in \Sigma_m/\mathrm{conj}$

$$\begin{array}{cc} \downarrow & \mapsto \\ l \in \mathbb{Z}/_{|\sigma|}^\times & [\sigma^l] \end{array}$$

$|\sigma| = \mathrm{lcm}(\text{length of cycles}) \Rightarrow \sigma^l$ has the same cycle decomp as σ.

$\Rightarrow \hat{\mathbb{Z}}^\times$ action is trivial on Σ_m/conj

Since $\hat{\mathbb{Z}}^\times = Gal(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$, we have

$$\begin{tikzcd} RU(\Sigma_m) \arrow[r] \arrow[dr, dashed] & Cl(\Sigma_m, \mathbb{Q}) \\ & Cl(\Sigma_m, \mathbb{Z}) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Relation to restriction and transfer

Given $\varphi : H \to G$ get a map $\varphi/\mathrm{conj} : H/\mathrm{conj} \longrightarrow G/\mathrm{conj}$

inducing $Cl(G, \mathbb{Q}(\mu_\infty)) \longrightarrow Cl(H, \mathbb{Q}(\mu_\infty))$

Fact: Commutative square

$$\begin{tikzcd} RU(G) \arrow[r, "\mathrm{Res}_\varphi"] \arrow[d] & RU(H) \arrow[d] \\ Cl(G, \mathbb{Q}(\mu_\infty)) \arrow[r, "\mathrm{Res}_{\varphi/\mathrm{conj}}"] & Cl(H, \mathbb{Q}(\mu_\infty)) \end{tikzcd}$$

Assume $H \leq G$, want $\mathrm{Tr}_H^G : Cl(H, \mathbb{Q}(\mu_\infty)) \longrightarrow Cl(G, \mathbb{Q}(\mu_\infty))$

$$\mathrm{Tr}_H^G(f)([g]) = \frac{1}{|H|} \sum_{\substack{l \in G \\ lgl^{-1} \in H}} f(lgl^{-1})$$

$$= \sum_{\substack{lH \in G/H \\ lgl^{-1} \in H}} f(lgl^{-1})$$

If $\varphi : H \twoheadrightarrow G$ then $\mathrm{Tr}_\varphi(f)([g]) = \frac{1}{|\ker \varphi|} \sum_{h \in \varphi^{-1}(g)} f(h)$.

<u>Power ops</u> $\quad V \circlearrowleft G \longmapsto V^{\otimes m} \circlearrowleft G^{\times m} \rtimes \Sigma_m = G \wr \Sigma_m$

Define $\mathbb{P}_m([V]) = [V^{\otimes m}] \in RU(G \wr \Sigma_m)$

Multiplicativity: $(V \otimes W)^{\otimes m} \cong V^{\otimes m} \otimes W^{\otimes m} \Rightarrow \mathbb{P}_m$ multiplicative

Additivity Fails:

As vectorspaces $\quad (V \oplus W)^{\otimes m} \cong \bigoplus_{i+j=m} \binom{m}{i} V^{\otimes i} \otimes W^{\otimes j}$

Failure of additivity controlled by transfer maps:

$$\mathbb{P}_m([V]+[W]) = \mathbb{P}_m([V]) + \sum_{\substack{i+j=m \\ i,j>0}} \mathrm{Tr}_{G\wr\Sigma_i \times G\wr\Sigma_j}^{G\wr\Sigma_m}\big(\mathbb{P}_i([V]) \boxtimes \mathbb{P}_j([W])\big)$$
$$+ \mathbb{P}_m([W])$$

Idea: $G = e \quad$ let $\underline{m} = \underline{i} \cup \underline{j}$

$$V^{\otimes \underline{i}} \otimes W^{\otimes \underline{j}} \hookrightarrow \bigoplus_{\substack{X \subseteq \underline{m} \\ |X| = i}} V^{\otimes X} \otimes W^{\otimes (\underline{m} \setminus X)} \qquad \Sigma_i \times \Sigma_j\text{-equivariant map}$$

Extend to $\mathbb{C}[\Sigma_m] \otimes_{\mathbb{C}[\Sigma_i \times \Sigma_j]} -$.

From G-reps to virtual G-reps.

$$\mathbb{P}_0([V]) = 1 \quad \text{so} \quad \mathbb{P}_0(-[V]) = 1.$$

$$\mathbb{P}_1([V]) = [V] \quad \text{so} \quad \mathbb{P}_1(-[V]) = -[V].$$

let $\mathrm{Tr}_{i,j} = \mathrm{Tr}_{G\wr\Sigma_i \times G\wr\Sigma_j}^{G\wr\Sigma_{i+j}}$

Use induction to define $\mathbb{P}_m$ on $RU(G)$.

pulls out

$$\text{Eg. } 0 = \mathbb{P}_2([V] + (-[V])) = \mathbb{P}_2([V]) + \mathrm{Tr}_{1,1}([V] \otimes (-[V])) + \mathbb{P}_2(-[V])$$

$$\text{so } \mathbb{P}_2(-[V]) = \mathrm{Tr}_{1,1}([V] \otimes [V]) - \mathbb{P}_2([V]).$$

Ex. $\mathbb{P}_2 : \mathbb{Z} \longrightarrow RU(\Sigma_2) \cong \mathbb{Z}[x]/_{x^2-1}$.

$$k \longmapsto \frac{k^2+k}{2} + \frac{k^2-k}{2} x.$$

Properties of the $\mathbb{P}_m$'s:

1) Special values: $\mathbb{P}_0(x) = 1$, $\mathbb{P}_1 = id$.

2) $\mathbb{P}_m(x+y) = \Sigma \ldots$

3) $\mathrm{Res}^{G\wr\Sigma_m}_{G\wr\Sigma_i \times G\wr\Sigma_j}(\mathbb{P}_m) = \mathbb{P}_i \otimes \mathbb{P}_j$.

Failure of additivity, let $I_m \subseteq RU(G\wr\Sigma_m)$ an ideal

be $\mathrm{im}\left(\bigoplus_{\substack{i+j=m \\ i,j>0}} \mathrm{Tr}_{i,j}\right)$, this controls additivity

so $\mathbb{P}_m/I_m : RU(G) \longrightarrow RU(G\wr\Sigma_m)/I_m$ is additive.

<u>Interaction w/ character map</u>

New ingredient: An integer partition of m $\lambda \vdash m$ is a function

$$\lambda : \mathbb{N}_{>0} \longrightarrow \mathbb{N} \quad \text{s.t.} \quad \sum_i \lambda_i \cdot i = m.$$

An integer partition of m decorated by G/conj is function

$$\lambda : \mathbb{N}_{>0} \times G/\mathrm{conj} \longrightarrow \mathbb{N}$$

$$\text{s.t.} \quad \sum_{i,[g]} \lambda_{i,[g]} \cdot i = m$$

set of parts of m deco by G/conj

Canonical bijection: $G \wr \Sigma_m /_{\mathrm{conj}} \cong \mathrm{Parts}(m, G/\mathrm{conj})$

Idea: If $\sigma = (1 \dots m)$ $\quad (g_1, \dots, g_m, \sigma) \sim (g_m \cdots g_1, e, \dots, e, \sigma)$

$$\begin{tikzcd}
RU(G) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\chi"'] & RU(G \wr \Sigma_m) \arrow[d] \\
Cl(G, \mathbb{Q}(\mu_m)) \arrow[r, dashed, "\mathbb{P}_m^{cl}"] & Cl(G \wr \Sigma_m, \mathbb{Q}(\mu_m))
\end{tikzcd}$$

Prop: $\mathbb{P}_m^{cl}(f)(\lambda) = \prod_{i,[g]} f([g])^{\lambda_{i,[g]}}$.

Sym powers and Adams ops from power ops:

Forgot: $RU(\Sigma_m)/_{I_m} \cong \mathbb{Z}$ $\qquad$ $\begin{tikzcd} RU(\Sigma_m) \arrow[r] \arrow[r, bend left, "\chi_{[(1 \dots m)]}"] & RU(\Sigma_m)/_{I_m} \cong \mathbb{Z} \end{tikzcd}$

Big: $\Delta : G \times \Sigma_m \longrightarrow G \wr \Sigma_m$

$$\begin{tikzcd}
 & & RU(G) \\
RU(G) \arrow[r, "\mathbb{P}_m"] \arrow[urr, bend left, "\beta_m : V \mapsto V^{\otimes m}/\Sigma_m"] \arrow[ddrr, bend right] & RU(G \wr \Sigma_m) \arrow[r, "\mathrm{Res}_\Delta"] & RU(G \times \Sigma_m) \arrow[u, "\mathrm{Tr}^G_{G \times \Sigma_m}"'] \\
 & & RU(G) \otimes RU(\Sigma_m) \arrow[u, phantom, "\cong" rotate=90] \arrow[d, "\mathrm{id} \otimes \chi_{[(1 \dots m)]}"] \\
 & & RU(G)
\end{tikzcd}$$

mth Adams op $\rightarrow$ Ψ_m additive

Formulas for everything:

- $(\mathrm{Res}_\Delta \mathbb{P}_m)(f)([g], \tau \vdash m) = \prod_i f([g^i])^{\tau_i}$
- $\psi_m(f)([g]) = f([g^m])$
- $\beta_m(f)([g]) = \frac{1}{m!} \sum_{\tau \vdash m} \frac{m!}{\prod_i (i^{\tau_i} \tau_i!)} \prod_i f([g^i])^{\tau_i}$

$$= \sum_{\tau \vdash m} \frac{1}{\prod_i \tau_i!} \prod_i \left(\frac{f([g^i])}{i} \right)^{\tau_i}.$$

Finally: Using these fmlas can check that

$$\sum_{m \geq 0} \beta_m t^m = \exp\left(\sum_{k > 0} \frac{\psi_k}{k} t^k \right)$$

Lives in $(\mathbb{Q} \otimes RU(G))[[t]]$

In fact lives here $RU(G)\langle\langle t \rangle\rangle \subseteq (\mathbb{Q} \otimes RU(G))[[t]]$.

<u>Burnside rings</u>

A G-set is a finite set X equipped w/ an action of G.

Given G-sets X, Y can form

$X \sqcup Y$ disjoint union

$X \times Y$ the product G-acts diagonally.

Use these to define $+, \times$

Let $A(G)$ be the Grothendieck ring of iso classes of G-sets under $+, \times$.

The building blocks for G-sets are transitive G-sets and $G/H \cong G/K$ iff H and K are conjugate.

Additively, $A(G)$ is the free ab gp on iso classes of transitive G-sets.

i.e. let $\mathrm{Conj}(G) = \mathrm{Sub}(G)/\mathrm{conj}$

then $A(G) \cong \bigoplus_{[H] \in \mathrm{Conj}(G)} \mathbb{Z}\{[G/H]\}$.

To understand the mult. struct. Need

$$G/H \times G/K \cong \coprod_{HgK \in H \backslash G/K} G/(H^g \cap K) \quad \text{as G-sets.}$$

Ex. $A(C_p)$ two iso classes of transitives

$$[C_p/C_p] = 1 \quad \text{and} \quad [C_p/e] = x$$

$$C_p/e \times C_p/e \cong \coprod_{e \backslash C_p/e} C_p/e \quad \text{so } x^2 = px$$

$$A(C_p) \cong \mathbb{Z}[x]/x^2 - px.$$

<u>Linearization</u>

$$L : A(G) \longrightarrow RU(G)$$
$$G \curvearrowright X \longmapsto G \curvearrowright \mathbb{C}\{X\}$$

send $\sqcup$ to $\oplus$ and $\times$ to $\otimes$ so a ring map.

Not nec inj or surj: Consider $G = C_2 \times C_2$.

<u>Restriction & transfers</u>

Given $\varphi : H \to G$

$\mathrm{Res}_\varphi : A(G) \longrightarrow A(H)$ ring map

$G \curvearrowright X \longmapsto H \overset{\varphi}{\curvearrowright} G \curvearrowright X$

$\mathrm{Tr}_\varphi : A(H) \longrightarrow A(G)$ additive

$H \curvearrowright X \longmapsto G \curvearrowright G \overset{\varphi}{\times}_H X := G \times X /_{(g,hx) \sim (g\varphi(h), x)}$

satisfy double coset formula & Frob reciprocity

Ex. $H \leq G$ $\mathrm{Tr}_H^G([H/K]) = [G \times_H H/K] = [G/K]$

Tr_H^G sends basis elts to basis elts.

$G \to e$ $\mathrm{Tr}_G^e([X]) = [e \times_G X] = [X/G] = |X/G| \in A(e)$

External mult: $\boxtimes : A(G) \otimes A(H) \longrightarrow A(G \times H)$

$G \curvearrowright X \quad H \curvearrowright Y \longmapsto G \times H \curvearrowright X \times Y$

Not generally an iso. [Consider $C_2 \times C_2$]

<u>Character theory</u>:

Let $\mathrm{Marks}(G, \mathbb{Z}) = \mathrm{Fun}(\mathrm{Conj}(G), \mathbb{Z})$

$\chi : A(G) \longrightarrow \mathrm{Marks}(G, \mathbb{Z})$

$G \curvearrowright X \longmapsto \chi([X])([H]) = |X^H|$

Fact: injective ring map and rationally an iso.

Formulas for res and tr on Marks:

For $H \leq G$, similar to RU story: $\mathrm{Tr}_H^G(f)([K]) = \sum_{\substack{gH \in G/H \\ K^g \leq H}} f([K^g])$

For surjs, the story is interesting.

$\mathrm{Tr}_G^e(f)(e) = \frac{1}{|G|}\sum_{g\in G} f([\langle g\rangle])$ Burnside's orbit counting lemma.

$$\mathrm{Marks}(G,\mathbb{Z}) \xrightarrow{L} \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty))$$

$$L(f)([g]) = f([\langle g\rangle]).$$

compatible w/ restriction and transfer.

<u>Power ops</u>

$$\mathbb{P}_m : A(G) \longrightarrow A(G\wr\Sigma_m)$$
$$G \curvearrowright X \longmapsto X^{\times m} \circlearrowleft G\wr\Sigma_m.$$

Something special happens:

$$\mathbb{P}_m([G/H]) = [G/H^{\times m}]$$

but $G/H^{\times m} \cong G\wr\Sigma_m / H\wr\Sigma_m$ as $G\wr\Sigma_m$-sets.

$\mathbb{P}_m$ sends basis elts to basis elts.

Ex. $G = e$ $\quad \mathbb{Z} \xrightarrow{\mathbb{P}_m} A(\Sigma_m)$

$$0 \longmapsto 0$$
$$[e/e] = 1 \longmapsto [\Sigma_m/\Sigma_m] = 1$$
$$1+1 = 2 \longmapsto \sum_{i+j=m} \mathrm{Tr}_{\Sigma_i\times\Sigma_j}^{\Sigma_m}([\Sigma_i\times\Sigma_j/\Sigma_i\times\Sigma_j])$$
$$= \sum_{i+j=m} [\Sigma_m/\Sigma_i\times\Sigma_j]$$

Continuing, see elts in the image of $\mathbb{P}_m$ are built out of Σ_m-sets of the form $[\Sigma_m/\Sigma_\lambda]$ $\lambda \vdash m$ (means $\sum_i \lambda_i \cdot i = m$) and $\Sigma_\lambda = \prod_i \Sigma_i^{\times \lambda_i}$.

Def: Let $\mathring{A}(G,m) \subseteq A(G \wr \Sigma_m)$ be the span of the smallest collection of basis elts s.t. the span contains $\mathbb{P}_m(A(G))$.

Equivalent constructions of $\mathring{A}(G,m)$

① Def: A $G \wr \Sigma_m$-set X is <u>submissive</u> if $X \hookrightarrow \underbrace{Y^{\times m}}_{G \wr \Sigma_m\text{-sets}}$ for some G-set Y.

Submissive $G \wr \Sigma_m$-sets are closed under $\sqcup, \times$.

$$(X \sqcup X' \hookrightarrow (Y \sqcup Y')^{\times m})$$

Fact: $\mathring{A}(G,m)$ is the Grothendieck ring of iso classes of submissive $G \wr \Sigma_m$-sets.

② Let $\mathrm{Parts}(G,m)$ be the set of $\mathrm{Conj}(G)$-decorated integer partitions of m. $\lambda : \mathrm{Conj}(G) \times \mathbb{N}_{>0} \longrightarrow \mathbb{N}$

$$\sum_{[H],i} \lambda_{[H],i} \cdot i = m.$$

$$\alpha : \mathrm{Parts}(G,m) \underset{\longrightarrow}{\overset{\beta}{\longleftarrow}} \mathrm{Conj}(G \wr \Sigma_m)$$

$$\lambda \longmapsto \Big[\prod_{[H],i} (H \wr \Sigma_i)^{\times \lambda_{[H],i}} \Big]$$

Fact: α admits a canonical retract β.

Let $\mathrm{Parks}(G,m) = \mathrm{Fun}(\mathrm{Parts}(G,m), \mathbb{Z})$

$$\begin{tikzcd}
\mathring{A}(G,m) \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner\, pb"] & A(G \wr \Sigma_m) \arrow[d, "\alpha"] \\
\mathrm{Parks}(G,m) \arrow[r, "\beta^*"] & \mathrm{Marks}(G \wr \Sigma_m, \mathbb{Z})
\end{tikzcd}$$

Win!

$$\begin{tikzcd} A(G) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\chi"] & \mathring{A}(G,m) \arrow[r, hook] \arrow[d] & A(G \wr \Sigma_m) \arrow[d] \\ \mathrm{Marks}(G) \arrow[r, dashed, squiggly, "\mathbb{P}_m^{\mathrm{Marks}}"] & \mathrm{PMarks}(G,m) \arrow[r, "\beta^*"] & \mathrm{Marks}(G \wr \Sigma_m) \end{tikzcd}$$

$$f \longmapsto \mathbb{P}_m^{\mathrm{Marks}}(f)(\lambda) = \prod_{[H],i} f([H])^{\lambda_{[H],i}}$$

Morava E-theory

p-adic K-thy : let $KU_p = KU_p^{\wedge}$

$RU(G)$ $\qquad G \curvearrowright V$

$$\begin{tikzcd} EG \times V \arrow[r] \arrow[d] & V \arrow[d] & \rightsquigarrow & EG \times_G V \arrow[d] \\ EG \arrow[r] & * & & BG \end{tikzcd}$$

$$\begin{tikzcd} RU(G) \arrow[d] \\ KU^0(BG) \cong RU(G)^{\wedge}_{I_{aug}} \arrow[d] \\ KU_p^0(BG) \cong RU(G)^{\wedge}_{(p)+I_{aug}} \end{tikzcd}$$

$I_{aug} = \ker(RU(G) \to \mathbb{Z})$

$KU^0(BG) \cong RU(G)^{\wedge}_{I_{aug}}$ ← poorly behaved as a $\mathbb{Z}$-module

$KU_p^0(BG) \cong RU(G)^{\wedge}_{(p)+I_{aug}} \cong \mathbb{Z}_p \otimes RU(G)/J$ ← free $\mathbb{Z}_p$-mod.

where $J = \ker(RU(G) \xrightarrow{res} RU(S))$ $\quad S \subseteq G$ sylow p.

$\mathrm{rk}_{\mathbb{Z}_p} KU_p^0(BG) = |G_p/\mathrm{conj}|$ $\quad G_p$ = set of p-power order elts in G $= \hom_{cts}(\mathbb{Z}_p, G)$

Ex. $KU_p^0(B\mathbb{Z}/p^k) \cong \mathbb{Z}_p[x]/x^{p^k}-1$.

$KU_p^0(BH) \cong \mathbb{Z}_p$. $\qquad p \nmid |H|$

Character thy (Adams) $\quad \chi : KU_p^0(BG) \longrightarrow Cl_p(G, \mathbb{Q}_p(\mu_{p^\infty}))$

$$Cl_p(G, \mathbb{Q}_p(\mu_{p^\infty})) = \mathrm{Fun}(G_p/\mathrm{conj}, \mathbb{Q}_p(\mu_{p^\infty}))$$

Idea: $$\begin{tikzcd} \mathbb{Z}_p \arrow[r, "\alpha"] \arrow[dr] & G \\ & \mathbb{Z}/p^k \arrow[u, dotted, "\alpha"'] \end{tikzcd}$$ then need $KU_p^0(BG) \longrightarrow \mathbb{Q}_p(\mu_{p^\infty})$

$$KU_p^0(BG) \xrightarrow{\alpha^*} KU_p^0(B\mathbb{Z}/p^k) \cong \mathbb{Z}_p[x]/_{x^{p^k}-1} \twoheadrightarrow \mathbb{Z}_p[x]/_{\Phi_{p^k}(x)} \hookrightarrow \mathbb{Q}_p(\mu_{p^\infty})$$

What is Morava E-thy?

let k be a perfect field of char p. and let $\Gamma/_k$ be a ht n fgl over k.

$x +_\Gamma y \in k[\![x,y]\!]$ satisfying
1) $x +_\Gamma 0 = x$
2) $(x +_\Gamma y) +_\Gamma z = x +_\Gamma (y +_\Gamma z)$
3) $x +_\Gamma y = y +_\Gamma x$.

Ht n means $[p]_\Gamma(x) = \underbrace{x +_\Gamma \cdots +_\Gamma x}_{p\text{-times}} = x^{p^n} + \cdots$

Ex. $x +_{\hat{\mathbb{G}}_m} y = x + y + xy$. $/\mathbb{F}_p$.

This data determines a $K(n)$-local E_∞-ring spectrum $E = E(\Gamma, k)$ w/ even periodic htpy gps

$$\pi_* E \cong W(k)[\![u_1, \ldots, u_{n-1}]\!][\beta^{\pm}] \quad |\beta| = 2.$$

$k = \mathbb{F}$ $W(\mathbb{F}_p) = \mathbb{Z}_p$.

$\pi_0 E$ = Lubin-Tate ring - which carries the universal deformation $\Gamma/_k$.

[Goerss-Hopkins-Miller]

Since E is a spectrum, the associated coh thy determines a global Mackey functor

Since E is a htpy com ring spectrum, assoc. coh. thy determines a global Green functor

Since E is an E_∞-ring, get a global power functor

Since E is $K(n)$-local get transfers along surjs.

Power ops: $BG \to E \rightsquigarrow BG^{\times m}_{h\Sigma_m} \longrightarrow E^{\wedge m}_{h\Sigma_m} \xrightarrow{\mu_m} E$

$$BG^{\times m}_{h\Sigma_m} \cong BG \wr \Sigma_m$$

gives $E^0(BG) \xrightarrow[\mathbb{P}_m]{} E^0(BG \wr \Sigma_m)$

$K(n)$-local transfers:

let $E_0^\wedge(BG) = \pi_0 L_{K(n)}(E \wedge BG)$

$K(n)$-local Tate vanishing $\Rightarrow$ canonical iso $E^0(BG) \cong E_0^\wedge(BG)$.

Given $\varphi : H \to G$, can

$$\mathrm{Tr}_\varphi : E^0(BH) \cong E_0^\wedge(BH) \xrightarrow{E_0^\wedge(B\varphi)} E_0^\wedge(BG) \cong E^0(BG).$$

Since E is even periodic $E^0(BS^1) \cong E^0[[x]]$

is the ring of fncs on a formal gp.

$$\mathrm{spf}\, E^0(BS^1) = \mathbb{G}_E : \mathrm{CompLocAlg}_{E^0} \longrightarrow \mathrm{Ab}$$

$$(R, m_R) \longmapsto (m_R, 0, +_{\mathbb{G}_E})$$

Ex. $k = \mathbb{F}_p$ $\Gamma = x + y + xy = x +_{\hat{\mathbb{G}}_m} y$

$$[p]_{\hat{\mathbb{G}}_m}(x) = x^p + \dots$$

$$E(\hat{\mathbb{G}}_m, \mathbb{F}_p) = KU_p^\wedge.$$

Prop: $\mathrm{spf}\, E^0(B\mathbb{Z}/p^k) \cong \mathbb{G}_E[p^k] \cong \mathrm{spf}\, E^0[[x]] / [p^k]_{\mathbb{G}_E}(x)$

HKR character theory:

The analogue of $\mathbb{Q}(\mu_\infty)$: A $\mathbb{Q}\otimes E^0$-alg $C_0 = \mathbb{Q}\otimes D_\infty$

$$\mathbb{G}_E[p^k]^{\times n} = \underline{\mathrm{Hom}}(C_{p^k}^{\times n}, \mathbb{G}_E) \supseteq \underline{\mathrm{Level}}(C_{p^k}^{\times n}, \mathbb{G}_E)$$

"injective $C_{p^k}^{\times n} \hookrightarrow \mathbb{G}_E$".

Ex: $\mathcal{O}_{\underline{\mathrm{Level}}(C_p, \mathbb{G}_E)} \cong E^0[[x]]/\langle p\rangle_{\mathbb{G}_E}(x)$

where $\langle p\rangle_{\mathbb{G}_E}(x) = \dfrac{[p]_{\mathbb{G}_E}(x)}{x}$.

$$\underline{\mathrm{Level}}(C_{p^k}^{\times n}, \mathbb{G}_E) \longrightarrow \underline{\mathrm{Level}}(C_{p^{k-1}}^{\times n}, \mathbb{G}_E)$$

$$\mathcal{O}_{\underline{\mathrm{Level}}} \longleftarrow \mathcal{O}_{\underline{\mathrm{Level}}}$$

$$D_\infty = \operatorname{colim}_k \mathcal{O}_{\underline{\mathrm{Level}}(C_{p^k}^{\times n}, \mathbb{G}_E)} = \mathcal{O}_{\underline{\mathrm{Level}}(C_{p^\infty}^{\times n}, \mathbb{G}_E)} \circlearrowleft GL_n(\mathbb{Z}_p)$$

$D_\infty^{GL_n(\mathbb{Z}_p)} = E^0$. $\quad C_0 = \mathbb{Q}\otimes D_\infty$.

$$\mathbb{Q}\otimes E^0(B\mathbb{Z}/p^{k\,\times n})/I_{tr} \cong \mathbb{Q}\otimes \mathcal{O}_{\underline{\mathrm{Level}}(C_{p^k}^{\times n}, \mathbb{G}_E)}$$

$$I_{tr} = \operatorname{im}\Big(\bigoplus_{A \subsetneq \mathbb{Z}/p^{k\,\times n}} \mathrm{Tr}_A^{\mathbb{Z}/p^{k\,\times n}}\Big)$$

Let $G_p^{[n]} = \hom_{cts}(\mathbb{Z}_p^{\times n}, G)/_{\mathrm{conj}}$.

$$\chi : E^0(BG) \longrightarrow Cl_{n,p}(G, C_0) = \mathrm{Fun}(G_p^{[n]}, C_0)$$

$$\begin{tikzcd}
\mathbb{Z}_p^{\times n} \arrow[r, "\alpha"] \arrow[d] & G \\
\mathbb{Z}/p^{k\,\times n} \arrow[ur, dotted, "?\alpha"'] &
\end{tikzcd}$$

$$E^0(BG) \xrightarrow{\alpha^*} E^0(B(\mathbb{Z}/p^k)^{\times n}) \longrightarrow \mathbb{Q}\otimes E^0(B(\mathbb{Z}/p^k)^{\times n})/I_{tr} \hookrightarrow C_0 .$$

Thm (HKR) χ is a ring map, inj, $E^0(BG)$ is torsion-free, and $C_0 \otimes_{E^0} \chi$ is an iso.

$$\mathbb{Q}\otimes\chi : \mathbb{Q}\otimes E^0(BG) \xrightarrow{\cong} Cl_{n,p}(G, C_0)^{GL_n(\mathbb{Z}_p)}$$

Thm (Strickland) let $I_{tr} \subseteq E^0(B\Sigma_m)$ be $\mathrm{im}\Big(\bigoplus_{\substack{i+j=m \\ i,j>0}} Tr_{\Sigma_i\times\Sigma_j}^{\Sigma_m}\Big)$, then $E^0(B\Sigma_m)/I_{tr}$ is a free E^0-module and

$$\mathrm{spf}\, E^0(B\Sigma_m)/I_{tr} \cong Sub_m(\mathbb{G}_E)$$

$E^0 \longrightarrow E^0(B\Sigma_m) \longrightarrow E^0(B\Sigma_m)/I_{tr}$ is a ring map.

AHS gave an algebro geom interpretation.

Big diagram:

```latex
\begin{tikzcd}
 & & E^0(BG) \\
E^0(BG) \arrow[r, "P_m"] \arrow[urr, bend left, "\beta_m"] & E^0(BG\wr\Sigma_m) \arrow[r] & E^0(BG\times B\Sigma_m) \arrow[u, "Tr^G_{G\times\Sigma_m}"'] \\
 & & E^0(BG)\otimes_{E^0} E^0(B\Sigma_m) \arrow[u, phantom, "\cong" rotate=90] \arrow[d] \\
 & & E^0(BG)\otimes_{E^0} E^0(B\Sigma_m)/I_{tr}
\end{tikzcd}
```

$$\begin{tikzcd}
T_{p^k} \arrow[drr, bend left] & & \arrow[d, "m=p^k\ 1\otimes \varepsilon l_A"] \\
& & E^0(BG)
\end{tikzcd}$$

$\uparrow$ p^kth Hecke operator

Ganter: $$\sum_{m\geq 0} \beta_m t^m = \exp\Big(\sum_{k\geq 0} \frac{T_{p^k}}{p^k} t^{p^k} \Big)$$

subgps of order p^{kn}

$$\begin{tikzcd}
G[p^k] \subseteq \mathbb{G}_E \arrow[d] \\
\operatorname{Spf} E^0
\end{tikzcd}$$

$$\begin{tikzcd}
E^0 \arrow[r, "P_{p^{kn}}"] \arrow[drrr, bend right] & E^0(B\Sigma_{p^{kn}}) \arrow[r] & E^0(B\Sigma_{p^{kn}})/I_{tr} \arrow[d, dashed] & \\
& & E^0 & \mathrm{Marks}(G,\mathbb{Z}) \arrow[d] \\
A(G) \arrow[urrr, bend left, "\psi_{p^k}"] \arrow[r] & E^0(BG) \arrow[rr] & & Cl_{n,p}(G,G_0)
\end{tikzcd}$$

$$G \curvearrowright X \longmapsto \chi(G \curvearrowright X)([\alpha]) = |X^{\operatorname{im}\alpha}|$$

Partition functors + Universal exponential relations jt w/ Mehrle + Rose.

Global Mackey functor:

$$M^*: \mathrm{Gps}^{op} \longrightarrow \mathrm{Ab} \qquad M_*: \mathrm{Gps}_{inj} \longrightarrow \mathrm{Ab}$$

$M^*(G) = M_*(G)$ and satisfy a double coset fmla.

Really need a $(2,1)$-category of Gps

$$\begin{tikzcd}
H \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & G \arrow[from=U, to=D, Rightarrow, "\sigma"]
\end{tikzcd} \qquad \sigma \in G \text{ s.t. } \sigma f \sigma^{-1} = g$$

Ab is a 1-cat $\Rightarrow$ conj maps go to the same map.

Exp rels: $$\sum_{m\geq 0} \beta_m t^m = \exp\Big(\sum_{k\geq 1} \frac{\psi_k}{k} t^k \Big)$$

$$RU(G)\langle\langle t\rangle\rangle \subseteq \left(\mathbb{Q}\otimes RU(G)\right)[[t]]$$

$$\underset{\substack{\uparrow \\ \displaystyle\sum_{i\geq 0} r_i \frac{t^i}{i!}}}{}$$

If R is a global power functor

$$\begin{tikzcd} R(e) \arrow[r, "P_m"] \arrow[rr, bend right] & R(\Sigma_m) \arrow[r] & R(\Sigma_m)/I_{tr} \end{tikzcd}$$

P_m/I_{tr} is additive

Goal: An exp reln

$$\sum_{n\geq 0} P_m t^m = \exp\left(\sum_{k\geq 1} P_k/I_{tr}\, t^k\right)$$

specializing to give the exp relns we've described.

<u>Partition functors</u>

A $(2,1)$-cat. of partitions P.

$\text{ob}\, P = \Lambda = (X, \sim_\Lambda)$

finite set

if $\Omega = (Y, \sim_\Omega)$ $\quad \Lambda \xrightarrow{f} \Omega$ is a bij. $X \xrightarrow[\cong]{f} Y$ st

$x \sim_\Lambda x' \Rightarrow f(x) \sim_\Omega f(x')$.

let Σ_Ω = autos of Ω that are the identity mod $\sim_\Omega$.

A 2-mor $\begin{tikzcd} \Lambda \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & \Omega \end{tikzcd}$ is an element in Σ_Ω s.t. $\sigma f = g$.

Ex. $\underline{m}$ = trivial or indiscrete on $\{1, \ldots, m\}$, $\quad \Sigma_{\underline{m}} = \Sigma_m$

but $\quad \Sigma_{\underline{i}\sqcup\underline{j}} = \Sigma_i \times \Sigma_j$

$\Lambda \leq \Omega$ if $X = Y$ and id_X is a map of partitions.

Def: A partition functor M is a pair

$$M_*: P \longrightarrow Ab, \quad M^*: P^{op} \longrightarrow Ab$$

s.t. $M_*(\Lambda) = M^*(\Lambda)$

and satisfying a double coset formula

Ex. Fix a gp G. Two functors

$$G \wr \Sigma_{(-)} : P \longrightarrow Gps$$

$$G \times \Sigma_{(-)} : P \longrightarrow Gps$$

Can restrict a global Mackey functor along these to get partition functors.

<u>The monad Div</u>

let M be a partition functor

let $I_\Omega = \mathrm{im}\left(\bigoplus_{\Gamma < \Omega} M_*(\Gamma < \Omega) \right) \subseteq M(\Omega)$

If $\Omega \leq \Lambda$ and $\sigma \in \Sigma_\Lambda$ then $\sigma\Omega \leq \Lambda$

inducing an iso $M(\Omega)/I_\Omega \xrightarrow{\cong} M(\sigma\Omega)/I_{\sigma\Omega}$.

let $\mathrm{Div}(M)(\Lambda) = \left(\bigoplus_{\Omega \leq \Lambda} M(\Omega)/I_\Omega \right)^{\Sigma_\Lambda} \supseteq M(\Lambda)/I_\Lambda$.

Prop: $\mathrm{Div}(M)$ is a partition functor and the functor Div is a monad.

Idea: $\eta : M(\Lambda) \longrightarrow \mathrm{Div}(M)(\Lambda)$

Many mult. structures.

Def: A partition power functor is a partition ring R equipped w/ mult. ops

$$P_m : R(\underline{1}) \longrightarrow R(\underline{m})$$

satisfying identities

$R = RU(G \times \Sigma_{(-)})$

Fact: Div is a monad on all of these cats.

Ex. The initial partition ring.

let $\mathring{A}(\Lambda) \subseteq A(\Sigma_\Lambda)$ be the subgp generated by elts of the form $[\Sigma_\Lambda / \Sigma_P]$ for $P \leq \Lambda$

Assembles into a partition ring.

This is the initial partition ring.

Surprise? $\mathring{A}(\Lambda) \hookrightarrow A(\Sigma_\Lambda) \xrightarrow{\iota} RU(\Sigma_\Lambda)$ is an iso !!

$\Rightarrow RU(\Sigma_{(-)})$ is the initial partition ring.

Further: $RU(\Sigma_\Lambda)/I_\Lambda \cong \mathbb{Z}$

so $\mathrm{Div}(RU(\Sigma_{(-)}))(\underline{m}) = \left(\bigoplus_{\Lambda \leq \underline{m}} \mathbb{Z} \right)^{\Sigma_m}$

$$\cong \bigoplus_{\lambda \vdash m} \mathbb{Z} = Cl(\Sigma_m, \mathbb{Z})$$

and $\eta : RU(\Sigma_m) \longrightarrow Div(RU(\Sigma_{(-)}))(\underline{m})$

is the character map.

Symmetric fncs in R

Given a lax sym mon partition functor R

form $$R[\Sigma] = \bigoplus_{m \geq 0} R(\underline{m})$$

Com gr. ring $$R(\underline{i}) \otimes R(\underline{j}) \xrightarrow{\boxtimes} R(\underline{i} \sqcup \underline{j}) \xrightarrow{Tr_{\underline{i} \sqcup \underline{j}}^{\underline{i+j}}} R(\underline{i+j})$$

Can form $$R[\![\Sigma]\!] = \prod_{m \geq 0} R(\underline{m})$$

Interested in $$R[\![\Sigma]\!] \longrightarrow Div(R)[\![\Sigma]\!] = R\langle\!\langle \Sigma \rangle\!\rangle.$$

If R is a partition power functor

$$\begin{tikzcd}
R(\underline{1}) \arrow[r, "P_m"] \arrow[dr, bend right, "P_m/I_{\underline{m}}"'] & R(\underline{m}) \arrow[d] \arrow[dr, bend left, "\eta"] \arrow[d, phantom, "\times", shift left=4] & \\
& R(\underline{m})/I_{\underline{m}} \arrow[r, phantom, "\subseteq_u"] & Div(R)(\underline{m})
\end{tikzcd}$$

Thm: Assume R is a partition power functor, then in $R\langle\!\langle \Sigma \rangle\!\rangle$ we have
$$\sum_{m \geq 0} P_m t^m = \exp\Big(\sum_{k \geq 1} P_k/I_{\underline{k}}\, t^k \Big)$$

Cor: If S is a lax sym mon Div-alg and we have $R \to S$,

then get
$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \\ \mathrm{Div}(R) \arrow[ur, dashed, "?"'] & \end{tikzcd}$$
and thus an exp rel over S.

Thm: If R is a sym mon partition ring + $R(\underline{m})/I_m$ is a flat $R(\underline{0})$-module for all $m \in \mathbb{N}$, then $\mathrm{Div}(R)[\varepsilon]$ is the divided power envelope of $R[\varepsilon]$.