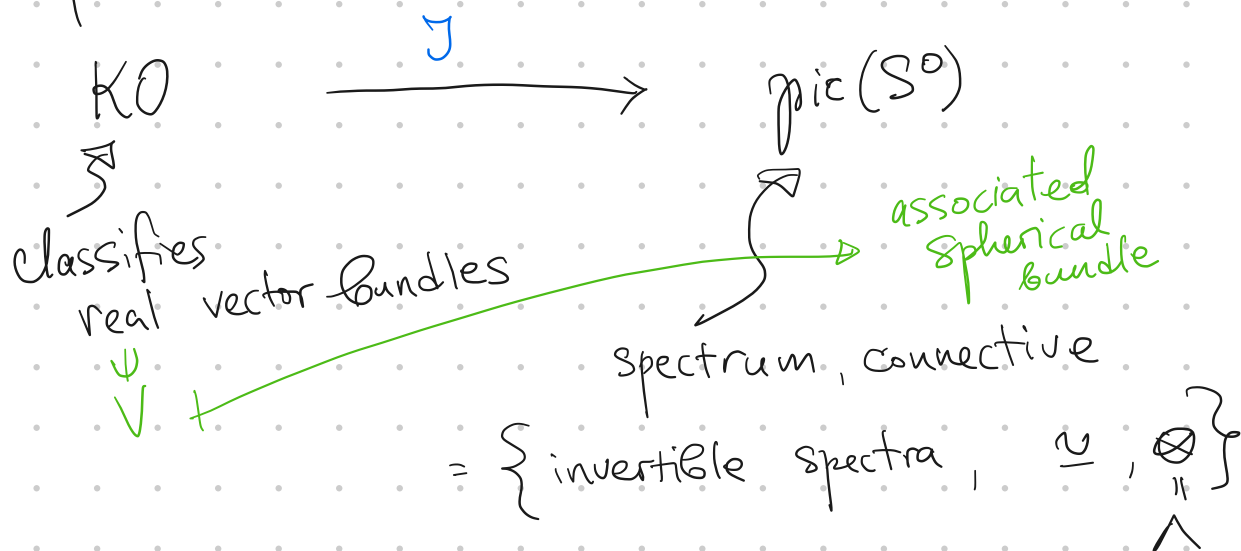


Representation spheres and the J-homomorphism

Hello!

J-homomorphism



$$\begin{aligned}\pi_0 \text{pic}(S^0) &= \mathbb{Z} = \{S^m, m \in \mathbb{Z}\} \\ \pi_1 \text{pic}(S^0) &= \mathbb{Z}/2 = \{\pm 1\} \subseteq \pi_0 S^0 \\ \pi_t \text{pic}(S^0) &= \pi_{t-1} S^0, \quad t \geq 2\end{aligned}$$

$$\Rightarrow \text{if } t \geq 2 \quad \pi_t J : \pi_t KO \longrightarrow \pi_{t-1} S^0$$

R ∞ ring spectrum $\rightsquigarrow$ sym. monoidal ∞ -category

$$\text{Pic}(R) \approx \left(\bigvee \text{Invertible (under } \bigwedge_R \text{) modules}, \cong \right)$$

Picard space, pointed at R
 $\pi_0 \text{Pic}(R) = \text{Pic}(R) = \underbrace{(\text{invertible } R\text{-modules})}_{\sim}$

Component of R in $\text{Pic}(R)$

Morphisms $R \rightarrow R$
in $\text{Pic}(R)$ $\cong$ Space of self-equivalences of R

$\text{haut}(R)$

$GL_1(R)$

(if $R = S^0$, $\text{haut}(R) = G$
old notation)

$\rightarrow$

$BGL_1(R)$

, $\pi_t BGL_1(R) = \pi_{t-1} GL_1(R)$
 $t \geq 1$

$GL_1(R) \hookrightarrow \Omega^\infty R$

$\downarrow$

$(\pi_0 R)^\times$

$\downarrow$

$\pi_0 R$

$\longrightarrow$

$\Rightarrow$ rest of
 $\pi_* \text{pic}(R)$

Representation spheres

G finite group

Σ real representation
ring of G

$[BG, KO]$

$\cong$

$RO(G)^\wedge_I$

Atiyah-Segal

$\neq \text{homog}$

$ko(KO)$

$I = \text{augmentation ideal}$

$$\begin{array}{ccc}
 I = \ker \longrightarrow & RO(G) & \longrightarrow \mathbb{Z} \\
 & V & \longmapsto \dim V
 \end{array}$$

eg. $G = C_2$ $RO(C_2) = \mathbb{Z}[\sigma] / (\sigma^2 - 1)$

$$\begin{array}{c}
 \downarrow \quad \uparrow \\
 \mathbb{Z}
 \end{array}$$

$$I = (\sigma - 1)$$

$$J : RO \longrightarrow \text{pic}(S^0)$$

$$\begin{array}{ccc}
 \rightsquigarrow & RO(G) \hat{I} & \longrightarrow (\text{spherical } G\text{-bundles}) \\
 & V & \longmapsto S^V
 \end{array}$$

Preview: $ko^{BG+} \longrightarrow \text{pic}(S^0)^{BG+} \longrightarrow \text{pic}(E)^{hG}$

on $\mathbb{A}_1$: $ko(G) \hat{I} \longrightarrow \text{Pic}(S^0 / E^{hG})$

$$\begin{array}{ccc}
 & & \\
 V & \longmapsto & (S^V \wedge E)^{hG}
 \end{array}$$

