Representation spheres and the J-homomorphism

Hello!

J - homomorphism

$$KO \xrightarrow{J} \mathrm{pic}(S^0)$$

classifies real vector bundles

$V \mapsto$ associated spherical bundle

spectrum, connective

$= \{\text{invertible spectra}, \simeq, \otimes\}$

$\otimes = \wedge$

$\pi_0\, \mathrm{pic}(S^0) = \mathbb{Z} = \{S^n, n\in\mathbb{Z}\}$

$\pi_1\, \mathrm{pic}(S^0) = \mathbb{Z}/2 = \{\pm 1\} \subseteq \pi_0 S^0$

$\pi_t\, \mathrm{pic}(S^0) = \pi_{t-1} S^0 \quad , \quad t \geq 2$

$\Rightarrow$ if $t \geq 2$ $\quad \pi_t J : \pi_t KO \longrightarrow \pi_{t-1} S^0$

R $\quad E_\infty$ ring spectrum $\rightsquigarrow$ sym. monoidal ∞-category

$\cup$

$\mathrm{Pic}(R) \simeq \left(\text{Invertible (under } \wedge_R\text{) modules}, \simeq\right)$

Picard space, pointed at R

$$\pi_0 \mathcal{P}ic(R) = \mathrm{Pic}(R) = (\underline{\text{invertible } R\text{-modules}}/\sim)$$

Component of R in $\mathcal{P}ic(R)$

Morphisms $R \to R$ in $\mathcal{P}ic(R)$

$$\simeq \underbrace{\text{Space of self-equivalences of } R}$$

$\mathrm{haut}(R)$

$GL_1(R)$

(if $R = S^0$, $\mathrm{haut}(R) = G$ old notation)

$$\rightsquigarrow \quad BGL_1(R) \quad , \quad \pi_t BGL_1(R) = \pi_{t-1} GL_1(R)$$

$t \geq 1$

$$\begin{tikzcd}
GL_1(R) \arrow[r, hook] \arrow[d] & \Omega^\infty R \arrow[d] \\
(\pi_0 R)^\times \arrow[r] & \pi_0 R
\end{tikzcd}$$

$\Rightarrow$ rest of $\pi_* \mathrm{Pic}(R)$

Representation spheres

G finite group

$$[BG, KO] \underset{\text{Atiyah-Segal}}{\cong} RO(G)^\wedge_I$$

real representation ring of G

$\pi(ko)^{hG} =$

$\pi_0(KO)^{hG}$

I = augmentation ideal

$$I = \ker \longrightarrow RO(G) \longrightarrow \mathbb{Z}$$
$$V \longmapsto \dim V$$

e.g. $G = C_2$ $\qquad RO(C_2) = \mathbb{Z}[\sigma]/(\sigma^2 - 1)$

$$\begin{array}{cc} \downarrow & \sigma \\ & \downarrow \\ \mathbb{Z} & 1 \end{array}$$

$I = (\sigma - 1)$

$$J : \quad KO \longrightarrow pic(S^0)$$

$$\rightsquigarrow \quad RO(G)^\wedge_I \longrightarrow \left(\text{spherical } G\text{-bundles}\right)$$
$$V \longmapsto S^V$$

Preview: $\quad KO^{BG_+} \longrightarrow pic(S^0)^{BG_+} \longrightarrow pic(E)^{hG}$

on π_0: $\quad RO(G)^\wedge_I \longrightarrow Pic(\text{sth} / E^{hG})$

$$V \longmapsto (S^V \wedge E)^{hG}$$