A particularly neat invertible $K(2)$-local spectrum at $p=2$

with Beaudry, Bobkova, Goerss, Henn, Pham
arXiv: 2212.07858

§ The usual suspects

Fix p prime, $n \geq 1$ height $\rightsquigarrow$

E : Lubin-Tate spectrum (for some formal group law of height n in char p.)

$E_n = E(\Gamma_n, k)$

$\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$: Morava stabilizer group

$\mathbb{S}_n \cong \mathrm{Aut}(\Gamma_n/k)$ = profinite group
cpt p-adic analytic grp of dim n^2.

$n=1$ $\quad \mathbb{S}_n = \mathbb{Z}_p^\times \circlearrowright E = E_1 = KU_p^\wedge$

§ $K(n)$-local Picard groups

$$\mathrm{Pic}_n := \mathrm{Pic}(Sp_{K(n)})$$

Hopkins, Hopkins-Mahowald-Sadofsky (1994)

$E_* = (\text{complete local ring in } \pi_0)[u^{\pm 1}]$

Recognition :

Lubin-Tate

$X \in \mathrm{Pic}_n \iff E_*X \simeq \Sigma^{\varepsilon} E_* \quad \varepsilon \in \{0,1\}$

ie $E_*X \in \mathrm{Pic}(E_*)$

Note: the action of $\mathbb{G}$ on E_*X may or may not be same as on E_*

$X \longmapsto E_*X \in \mathrm{Pic}(\text{Morava modules})$ or $E_*[\![\mathbb{G}]\!]$-modules

$$0 \to \kappa_n \to \mathrm{Pic}_n \to \underbrace{H^1(\mathbb{G}_n, E_*^{\times})}_{\mathrm{Pic}_n^{alg}}$$

κ_n = exotic Picard group

$H^0(\mathbb{G}, E_0X) \ni i_X \longleftarrow \uparrow$

if $X \in \kappa_n$, $\underline{\underline{E_*X}} \underset{\mathbb{G}}{\simeq} \underline{\underline{E_*S^0}}$

Adams-Novikov

$$i_X \in \underbrace{H^s(\mathbb{G}, E_tX)}_{\cong\, H^s(\mathbb{G}, E_tS^0)} \Rightarrow \pi_{t-s}X$$

determining the iso

i_X generates $H^*(\mathbb{G}, E_*X)$ as a module over $H^*(\mathbb{G}, E_*S^0)$

§ Examples

* $\kappa_n = 0$ if $2p-2 > n^2$ and $(p-1) \nmid n$

* $\kappa_1 = \begin{cases} 0 & \text{if } p \geq 3 \\ \mathbb{Z}/2 & \text{if } p = 2 \end{cases}$

HMS

20 years

$$\ast \quad K_2 = \begin{cases} 0 & \text{if } p \geq 5 \\ \mathbb{Z}/3 \times \mathbb{Z}/3 & \text{if } p = 3 \\ (\mathbb{Z}/8)^2 \times (\mathbb{Z}/2)^3 & \text{if } p = 2 \end{cases}$$

Goerss-Henn-Mahowald-Rezk

8+ years

Beaudry-Bobkova-Goerss-Henn-Pham-S.

Today Focus on just one $\mathbb{Z}/2 \subseteq K_2$

$\boxed{p=2}$

§ The construction

$\{\pm 1\} \subseteq E_0/2$

$\chi \in H^1(\mathbb{G}_n, \mathbb{Z}/2) \to H^1(\mathbb{G}_n, E_0/2)$

$$\begin{tikzcd}
 & & \mu_2 \cong C_2 \cong \mathbb{Z}/2 \\
\mathbb{G}_n \arrow[r, "\det"] \arrow[urr, bend left, "\chi"] \arrow[drr, bend right, "\overline{\det}"'] & \mathbb{Z}_2^\times \cong \mu_2 \times (1+4\mathbb{Z}_2) \arrow[ur, "\mathrm{proj}_1"] \arrow[dr, "\mathrm{proj}_2"'] & \\
 & & \mathbb{Z}_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathbb{Z}_p^\times \arrow[dr] & \\
\mathbb{S}_n \hookrightarrow GL_n(\mathbb{Z}_p(\zeta_{p^n-1})) \arrow[ur] \arrow[r, "\det"] & & \mathbb{Z}_p(\zeta_{p^n-1})^\times
\end{tikzcd}$$

Some subgroups:

$\ast \quad \mathbb{G}_n^1 := \ker(\overline{\det})$

* At $n=2$

$G_{48} :=$ maximal finite subgroup $\subseteq \ker(\det)$

$$\begin{tikzcd}
V \arrow[rr, mapsto] & & (E \wedge S^V) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
RO(C_2) \arrow[rr, "J_\chi"] & & \mathrm{Pic}_n \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
I = (\sigma - 1) \arrow[rr] & & \mathrm{Pic}_n^0 = \{X \in \mathrm{Pic}_n \mid E_*X \cong E_*\} \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
I^2 = (2\sigma - 2) \arrow[rr] & & \underline{\mathcal{K}_n}
\end{tikzcd}$$

standard act

act via χ G_n

Definition: $Q_n = J_\chi(2\sigma - 2) = (\underbrace{E \wedge S^{2\sigma-2}})^{hG_n}$

Example $n=1$ $\quad Q_1$ generates $\mathcal{K}_1 \simeq \mathbb{Z}/2$

$$G_1 \underset{\det}{\simeq} \mathbb{Z}_2^\times \simeq \mu_2 \times (1 + 4\mathbb{Z}_2)$$

$$\rightsquigarrow \quad Q_1 \simeq (E \wedge S^{2\sigma-2})^{h\mu_2 \times \mathbb{Z}_2}$$

Theorem (BBGHPS)

Q_2 is a non-trivial element of κ_2 of order 2.

Question Is the same true for all n?

$\boxed{n=2}$

§ Proof sketch

Homotopy fixed point spectral sequences
$V \in I^2 \subseteq RO(C_2)$

$$\begin{gathered}
\pi_{t-s}(E \wedge S^V)^{h\mathbb{G}_2} \\
\Uparrow \\
H^s(\mathbb{G}_2, E_t S^V) \\
\Psi \\
i_V
\end{gathered}$$

$d_r(i_V) \neq 0$??

Recognition: $J_\chi(V)$ is non-trivial

$\iff d_r(i_V) \neq 0$

$\tilde{\chi}$ = Bockstein on $\chi \in H^1(E_0/2)$

$$\downarrow \beta$$

$H^2(E_0)$

We show: $\underline{d_3(i_{2\sigma-2})} = \eta\, \tilde{\chi}\, i_{2\sigma-2} \neq 0$

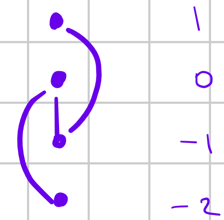
& $d_r(i_{4\sigma-4}) = 0$ for all r.

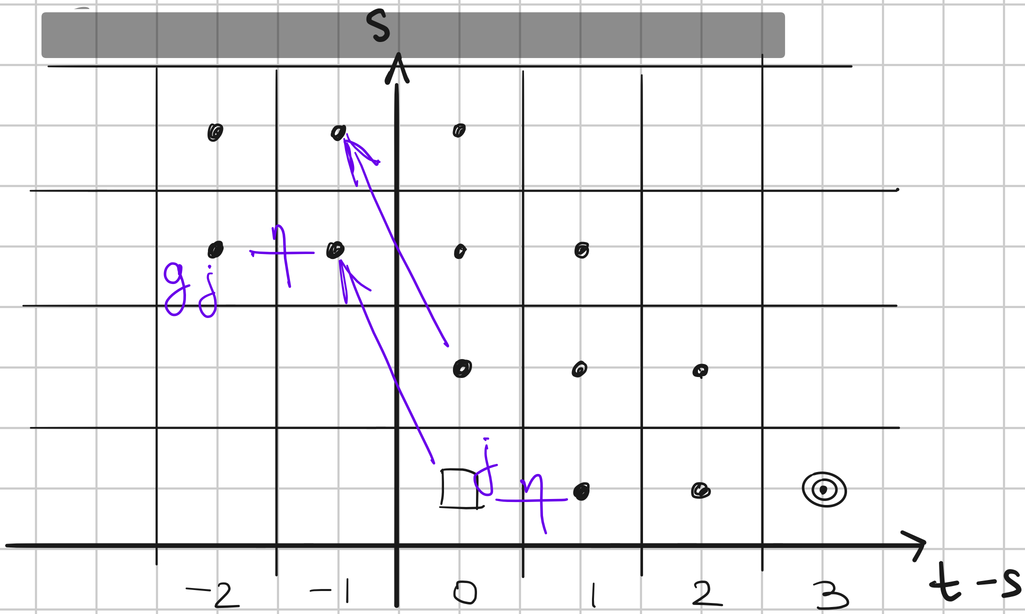
$$\begin{tikzcd}
\pi_{t-s} \mathrm{Tot}_3 (S^V)^{hC_2} \arrow[r] & \pi_{t-s} \mathrm{Tot}_3 (E \wedge S^V)^{hG_2} \\
s \leq 3 \quad H^s(C_2, \pi_t S^V) \arrow[u, Rightarrow] \arrow[r] & H^s(G_2, E_t S^V), \quad s \leq 3 \arrow[u, Rightarrow] \\
j_V \arrow[u, phantom, "\Psi"] \arrow[r, mapsto] & i_V \arrow[u, phantom, "\Psi"]
\end{tikzcd}$$

$$(S^{m\sigma - m})^{hC_2} \simeq \Sigma^{1-m} \mathbb{RP}^{m-1}_{-\infty} \simeq \Sigma^{-m} D\, Th(-m\xi)$$

3-Truncated spectral sequences

$$Tot_3(S^{2\sigma-2})^{hC_2} \simeq \Sigma^{-1}\mathbb{RP}^1_{-2}$$





$g \in H^2(C_2, \pi_0 S^{\sigma 0})$

$\downarrow$

$\tilde{\chi}$

$$d_2(j_{2\sigma-2}) = g\eta j_{2\sigma-2}$$

$\rightsquigarrow$ gj is a permanent cycle, with

$[gj] \neq 0 \in \pi_{-2} Tot_3(S^{2\sigma-2})^{hC_2}$

but $\eta[gj] = 0 \in \pi_{-1} Tot_3(S^{2\sigma-2})^{hC_2}$

Under our spectral sequence map

$$g j_{2\sigma-2} \longmapsto \tilde{\chi} i_{2\sigma-2}, \quad \tilde{\chi} = \text{Bockstein}$$

on $\chi \in H^1(G_2, \mathbb{Z}/2)$

$\downarrow$

$H^1(G_2, E_0/2)$

$[\tilde{\chi} i_{2\sigma-2}] \neq 0 \in \pi_{-2} \mathrm{Tot}(Q_2)$

(for degree reasons)

Cool fact

$$\begin{array}{ccc} \eta[gj] & \longmapsto & \text{class detected by} \\ & & \eta\tilde{\chi} i_{2\sigma-2} \in H^3(G_2, E_2 S^{2\sigma-2}) \\ \| & & \\ 0 & \longmapsto & 0 \end{array}$$

$\Rightarrow$ $\eta\tilde{\chi} i_{2\sigma-2}$ is hit by a differential

Only $d_3(i_{2\sigma-2}) = \eta\tilde{\chi} i_{2\sigma-2}$ is possible.

$\Rightarrow \quad Q_2$ is non-trivial.

Proof that $2Q_2 = 0$ is similar but much more involved.

$\square$