A Reading Course on Stable Homotopy Theory

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This self-guided reading course is an introduction to stable homotopy theory. Stable homotopy theory is a growing and diverse field with connections to algebraic and differential geometry, number theory, representation theory, and even combinatorics.

The resources included here are intended to help you organize your own reading course on stable homotopy theory. These materials, consisting of problem sets, solutions, lecture outlines, and detailed references, are adapted from a course taught by David Mehrle through the electronic Computational Homotopy Theory online research community in Spring 2024 and Spring 2025.

Learning Outcomes

After taking this course, students will be able to engage in research projects in stable homotopy theory or related fields that use stable techniques. Students will also gain experience communicating mathematics by giving talks, writing, and discussing mathematics with peers.

Prerequisites

- Students should have taken the equivalent of two semesters of algebraic topology at the graduate level, including homology, cohomology, and homotopy groups π_n for $n > 1$. Exposure to K-theory, characteristic classes, cobordism, and other topics in algebraic topology is not required, although it may be helpful.
- Students should have a working knowledge of category theory, including but not limited to: natural transformations, limits, colimits, adjunctions, abelian categories, and symmetric monoidal categories.

Topics

The course covers the following topics in the following order:

(1) Homotopical Categories
(2) Homotopy Limits and Colimits
(3) Stable Phenomena in Spaces
(4) Generalized Cohomology Theories
(5) Sequential Spectra
(6) Stability Theorems
(7) Cellular and Finite Spectra
(8) The Stable Homotopy Category
(9) The Smash Product
(10) Symmetric and Orthogonal Spectra
(11) Ring Spectra
(12) Operads
(13) Localization
(14) Chromatic Homotopy

Omissions

This course barely scratches the surface of modern stable homotopy theory. The most notable omission is spectral sequences and computations, which could be a whole class by itself. Other notable omissions include simplicial homotopy theory, model categories, ∞-categories, equivariant homotopy theory, ...

SHORTENING THE COURSE

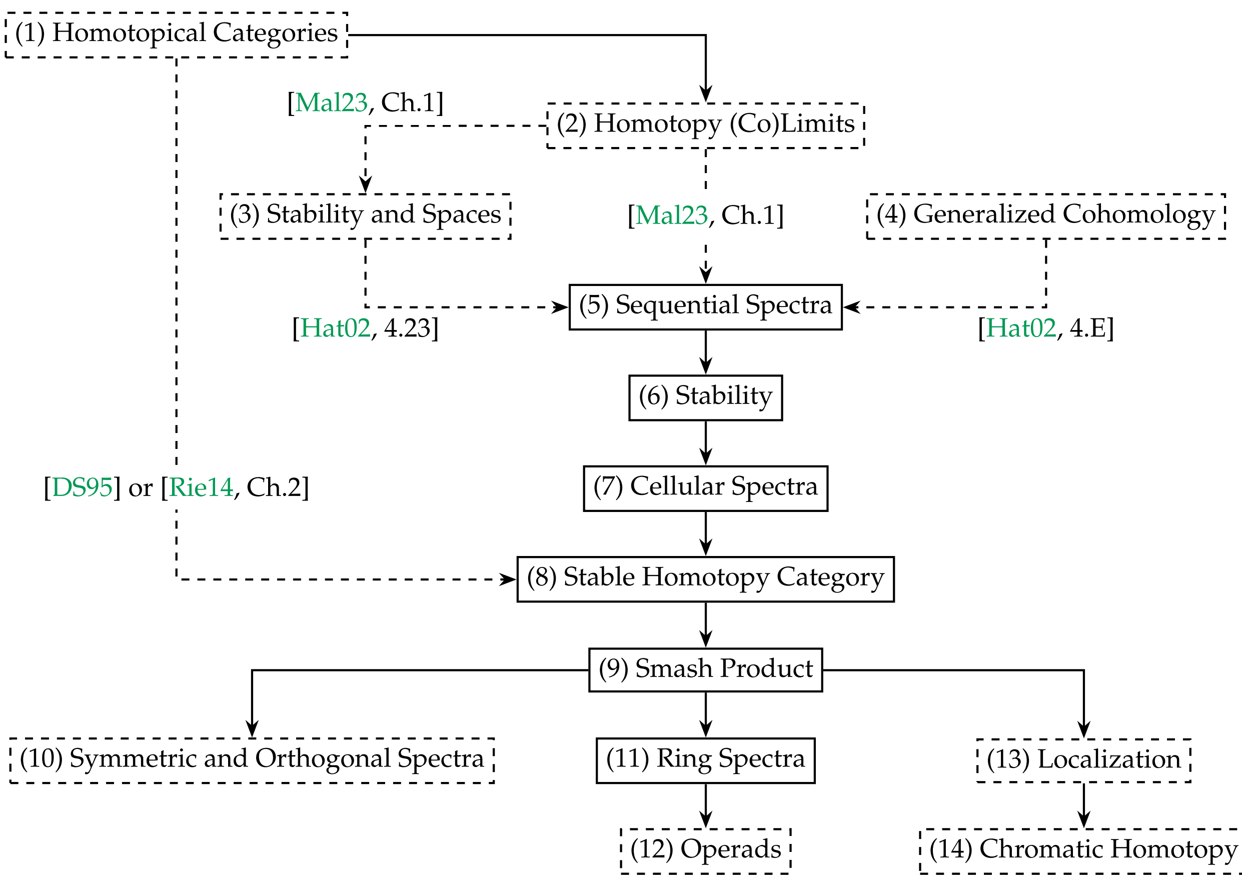
Topics (5)-(9) and (11) form the core of the course and should not be omitted. With a little background reading, sufficiently motivated readers may be able to jump in right at topic (5) and work their way through (11), skipping (10).

Topics (3) and (4) provide some motivation for future topics, if students have not yet seen Freudenthal's Suspension Theorem or Brown Representability.

Topics (1) and (2) provide a categorical framework for talking about homotopy theory. These topics were originally included in the course as a shortcut for bypassing model categories on the way to homotopy limits and colimits. However, if readers are interested primarily in homotopy limits and colimits in the category of spaces (or spectra), you might want to skip (1) and (2) and read [Mal23, Chapter 1] instead. Another option would be to keep Topics (1) and (2), but move them between topics (7) and (8). Alternatively, readers who want a more thorough introduction to the categorical aspects of modern homotopy theory should read the excellent [DS95].

Topic (9) deals with the smash product of spectra as a black box, simply providing categorical properties and discussing its interactions with common functors. For most mathematicians working with stable homotopy theory, this is more than sufficient. Topic (10) opens up the black box and discusses two ways to construct the smash product, although neither are the most modern approach. It can be safely skipped.

Topics (12), (13), and (14) are extra topics that can be omitted or swapped out for others. Topic (12) builds on topic (11) and is independent of the other two, but topic (14) requires (13) as a prerequisite.



A dependency tree for the topics in the course. Topics with solid borders form the core of the course; topics with dashed borders are optional. Solid arrows indicate mandatory prerequisites; dashed arrows indicate soft prerequisites that might be bypassed with other outside reading (suggested reading labels the arrow).

Logistics

The materials for this course consist of lecture outlines and problem sets (with solutions). The recommended structure for this reading course is to meet twice per week, tackling one topic per week. In the first meeting, one student should use the lecture outline to prepare and deliver a lecture. In the second meeting, all participants should work on the problem sets together. A suggested schedule is displayed below.

Week 1: Topic A	Lecture	(Meeting 1)
	Problem Session	(Meeting 2)
Week 2: Topic B	Lecture	(Meeting 2)
	Problem Session	(Meeting 3)
Week 3: Topic C	Lecture	(Meeting 4)
	Problem Session	(Meeting 5)
	$\vdots$	

Lecture Outlines

Each topic comes with a lecture outline detailing one possible path through the topic. The outlines suggest what to talk about, but they do not always contain proofs or detailed examples. Students will have to work to construct their own talk from the outlines provided.

These outlines are only suggestions. There are many different directions that one could go with each talk. Students should use the talks as an excuse to learn something they're curious about. The outlines might also be too much material for the time you have allocated. In that case, feel free to focus on the important things and cut other material.

Problem Sets and Solutions

It is recommended that students not only solve the problems, but also write their solutions down, before reading the solutions. For a self-guided reading course, we recommend that students engage in peer grading of each other's written solutions to the problem sets. Writing clearly and communicating ideas to your peers is a core skill of mathematical research!

A final exam is also provided, should students want to challenge themselves at the end of the semester.

Feedback, Questions, Comments, and Corrections

We hope that you find these resources useful! If you have any questions, comments, or feedback, or if you find errors that you would like to see corrected, please email David Mehrle at davidm@uky.edu.

References

The references I consulted while constructing this course are listed below. Almost all of the material referenced is available for free online; students should not need to buy any resources to engage with this course.

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