The chromatic Nullstellensatz

joint w/ Robert Burklund + Tomer Schlank

Thm (Hilbert's Nullstellensatz)

k alg closed field

$f_1, \ldots, f_n \in k[x_1, \ldots, x_m]$ such that $I = (f_1, \ldots, f_n) \neq (1)$

Then $f_1, \ldots, f_n$ have a common root.

$$\begin{tikzcd} k \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & A = \dfrac{k[x_1, \ldots, x_m]}{f_1, \ldots, f_n} \arrow[r, dashed] & k \end{tikzcd}$$

"unit map of any f.g. algebra $\neq 0$ admits a retract."

Defn: $\mathcal{C}$ sym mon ∞-category

A commutative algebra $k \in \mathrm{CAlg}(\mathcal{C})$ satisfies Nullstellensatz

if, for any compact non-terminal $A \in \mathrm{CAlg}_k(\mathcal{C})$,

the unit map $k \to A$ admits a retract.

Lubin–Tate theories: (Morava, Goerss–Hopkins–Miller, Lurie)

k perfect field of char p

$\mathbb{G}_0$ formal gp of height $n \geq 1$ $/k$.

$\Rightarrow$ $E_n(k, \mathbb{G}_0)$ $K(n)$-local E_∞ ring spectrum

When $k = \bar{k}$, $E_n = E_n(k)$.

$$\pi_* E_n(k) = W(k)[[\overset{0}{u_1}, \ldots, \overset{0}{u_{n-1}}]][\overset{2}{u}^{\pm 1}]$$

(Rognes)

Baker–Richter: $E_n(\bar{\mathbb{F}}_p)$ admits no nontrivial connected Galois extensions.

Thm (Burklund – Schlank – Y.)

$A \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ satisfies Nullstellensatz

$\Leftrightarrow$ $A \simeq E(L)$ L algebraically closed.

Proof cartoon for Hilbert's thm

$$k \longrightarrow A = \frac{k[x_1, \dots x_m]}{(f_1, \dots, f_n)} \dashrightarrow k$$

1) Choose $J \subseteq A$ max'l ideal

$$k \longrightarrow A \longrightarrow A/J \simeq k' \quad \text{some field}$$

2) Fact: field extensions which are fin. gen. as k-alg are finite $(k' = k)$.

Chromatic setting: $E_n(k) \longrightarrow A \dashrightarrow E_n(k')$

1) is nontrivial chromatically: can't quotient

Ex: $A \in \mathrm{CAlg}(\mathrm{Sp}_{K(1)})$

$\pi_0 A$ is a p-typical λ-ring / θ-ring.

$\theta : \pi_0 A \longrightarrow \pi_0 A$

Thm (BSY)

$A \in \mathrm{CAlg}(Sp_{T(n)})$ $\quad A \neq 0$

Then there exists an $\mathbb{E}_\infty$-ring map

$$A \longrightarrow E_n(L) \qquad L \text{ alg closed.}$$

Toy model $\quad R$ p-complete θ-ring

? $\quad R \dashrightarrow W(L) \qquad L$ alg closed

$$\begin{tikzcd} R \arrow[r, dashed, "\theta\text{-ring}"] \arrow[dr, "\substack{\text{ring}\\ \text{map}}"'] & W(L) \arrow[d] & \underline{\text{cofree}} \\ & L & \text{alg closed} \end{tikzcd}$$

Pf of thm uses higher height analogue of cofreeness of $W(-)$ (indep. due to Rezk).

Chromatic support for $\mathbb{E}_\infty$-rings

R p-local ring spectrum

Defn: $\mathrm{supp}(R) = \{ n \in \mathbb{N} \mid L_{T(n)} R \neq 0 \}$

$\Leftrightarrow T(n) \otimes R \neq 0$

$\Leftrightarrow K(n) \otimes R \neq 0$

Highly constrained for $\mathbb{E}_\infty$- rings.

$$R \in \mathrm{CAlg}(\mathrm{Sp})$$

<u>Thm</u> [May nilpotence] (Mathew-Naumann-Noel)

$$\mathrm{supp}(R) = \emptyset \quad \text{or} \quad 0 \in \mathrm{supp}(R).$$

<u>Thm</u> (Hahn)

$$\mathrm{supp}(R) = [0, n] \quad \text{for} \quad -1 \le n \le \infty$$

<u>Defn</u>: $n := \mathrm{ht}(R)$ height of R.

$\mathrm{ht}(R)$	R
-1	$\mathbb{F}_p$
0	$\mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}$
1	KU/KO, ku/ko
2	tmf, E_2, ...

<u>Alternate pf of Hahn's thm</u>

Suppose $L_{T(n)} R \neq 0$, WTS $L_{T(n-1)} R \neq 0$.

$$R \longrightarrow L_{T(n)} R \xrightarrow[\text{Thm}]{} \underline{E_n(L)} \qquad L \text{ alg closed.}$$

$\Rightarrow$ ring map $L_{T(n-1)} R \longrightarrow L_{T(n-1)} E_n(L) \neq 0$. □

<u>Redshift in alg K-thy</u>

$R \in \mathrm{CAlg}(\mathrm{Sp}) \rightsquigarrow K(R) \in \mathrm{CAlg}(\mathrm{Sp})$

Ausoni - Rognes: K(connective topological K-thy) was "of height 2"

Weak form of Ausoni - Rognes "chromatic redshift" conj.

$R \in \mathrm{CAlg}(\mathrm{Sp})$: $\mathrm{ht}\, K(R) = \mathrm{ht}\, R + 1$.

Thm (Land - Mathew - Meier - Tamme
Clausen - Mathew - Naumann - Noel)

$$\mathrm{ht}\, K(R) \leq \mathrm{ht}\, R + 1.$$

Thm (Hahn-Wilson)

$$L_{T(n+1)} K(BP\langle n \rangle) \neq 0.$$

Thm (Y.) $\mathrm{ht}(K(E_n)) = n+1$.

Cor (BSY) $R \in \mathrm{CAlg}(\mathrm{Sp})$, $\mathrm{ht}\, K(R) = \mathrm{ht}\, R + 1$.

Pf: $\mathrm{ht}(R) = n$

$$R \to L_{T(n)} R \xrightarrow[\text{Thm}]{} E_n(L).$$

$\Rightarrow$ ring map $\underset{\neq 0}{L_{T(n+1)} K(R)} \longrightarrow L_{T(n+1)} K(E_n(L)) \neq 0$. $\square$

Orientations of E_n

Note: $L_{K(n)} MU \neq 0$ so

thm: $MU \rightsquigarrow E_n(L)$ for some (turns out, any)
(CAlg map) alg closed L.

$\boxed{\text{Fix } k \text{ alg closed field.}}$

Thm [Universal orientability]

map of conn sp. $f: X \to \mathrm{pic}\, E_n(k)$

($\Omega^\infty X \to \mathrm{Pic}\, E_n(k)$
bundle of $E_n(k)$'s on $\Omega^\infty X$)

$\exists$ $\mathrm{CAlg}_{E_n(k)}$ map $MX \to E_n(k)$ $\Leftarrow$ $L_{K(n)} MX \neq 0$.

(f is null homotopic)

($MX \in \mathrm{CAlg}_{E_n(k)}$ Thom spectrum)

Ex: $f: \underbrace{BU}_{X} \xrightarrow{J} \mathrm{pic}\, \mathbb{S} \longrightarrow \mathrm{pic}\, E_n(k)$

$MX = MU \otimes E_n(k)$.

Thm (Hopkins-Lurie, Rezk ht=2)

A finite abelian p-group.

$$\mathrm{Hom}_{Sp}(A, \mathrm{pic}\, E_n(k)) = \Sigma^{n+1} A^*.$$

In particular,

$$\mu_p(E_n(k)) := \mathrm{Hom}_{Sp}(C_p, gl_1 E_n(k)) = \Sigma^n C_p .$$

Cor: $\underbrace{F}_{\text{discrepancy spectrum}} \to gl_1 E(k) \longrightarrow \overbrace{L_n\, gl_1 E(k)}^{\text{study via Rezk log}}$

Then $\tau_{\geq 0} F \simeq \tau_{\geq 0} \Sigma^n I_{\mathbb{Q}_p/\mathbb{Z}_p} \oplus (k^\times)^{tors}$

F studied
Ando – Hopkins – Rezk
F torsion, n-truncated.

Thm (Barthel · Carmeli – Schlank – Yanovski)
[Chromatic Fourier Transform]

M spectrum $\pi_* M = \begin{cases} 0 & * \notin [0,n) \\ \text{finite ab } p\text{-group} & \text{else} \end{cases}$

Let $M^* := \mathrm{Hom}(M, \Sigma^n I_{\mathbb{Q}_p/\mathbb{Z}_p})$.

Then there's a functorial equivalence of bicomm. bialg.

$$E_n(k)[\Omega^\infty M] \simeq E_n(k)^{\Omega^\infty M^*} .$$

Ex: $KU_p^{BA} \simeq KU_p^\wedge [A^*]$.